PROJECTED WRITTEN NOTES FROM THE M325K LECTURE ON TUESDAY, MARCH 26, 2024, ON SEC. 8.3 – EQUIVALENCE RELATIONS and Equivalence Classes and Proving Theorems about Equivalence Classes.

CLASS #19

## Recall the Definitions of the Relation Properties

Let $R$ be a relation on a set $A$.

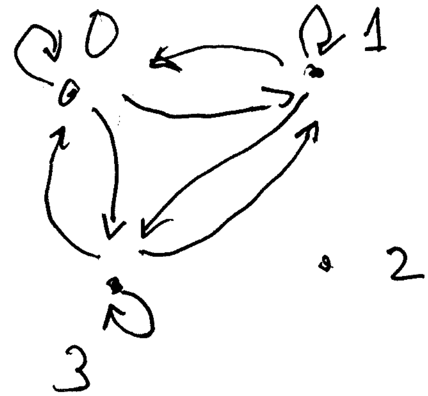
$R$ is Reflexive $\iff \forall x \in A,\ xRx$.

$R$ is Symmetric $\iff \forall x, y \in A$, if $xRy$, then $yRx$.

$R$ is Transitive $\iff \forall x, y, z \in A$, if $xRy$ and $yRz$, then $xRz$.

Ex: Let $A = \{0, 1, 2, 3\}$

Define relation $S$ on $A$ as the relation with the following directed graph:




$S$ is not reflexive

$S$ is Symmetric

$S$ is Transitive

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A homework problem, Problem #39 of Sec 8.3, has a false proof that "Every relation that is both symmetric and transitive must also be a reflexive."

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<u>Definition</u>: Let $A$ be a set and let $R$ be a relation on $A$.

$R$ is an <u>EQUIVALENCE RELATION</u>

$\Leftrightarrow$ $R$ is
① Reflexive
② Symmetric
③ Transitive.

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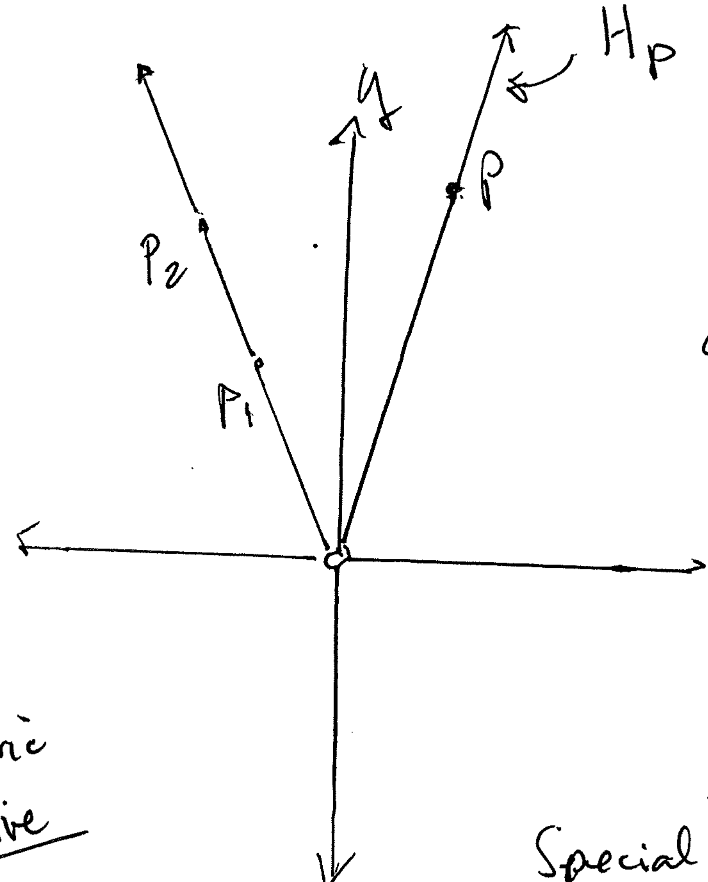
Sec 8.2 (t-2)

30. Let $A$ be the "punctured plane"; that is, $A$ is the set of all points in the Cartesian plane except the origin $(0, 0)$. A relation $R$ is defined on $A$ as follows: For all $p_1$ and $p_2$ in $A$, $p_1 \; R \; p_2 \Leftrightarrow p_1$ and $p_2$ lie on the same half line emanating from the origin.




$P_1 \, R \, P_2$ because $P_1$ and $P_2$ are on the same half-line emanating from the origin.

R is Reflexive
R is Symmetric
R is Transitive

So, R is an Equivalence relation.

$[P_1] = \{ \text{all } t \in A \text{ such that } t \, R \, P_1 \}$

$P_2 \in [P_1]$
$P_1 \in [P_1]$
$[P_1] = H_{P_1} = H_{P_2} = [P_2]$

Special Notation Here:

Let $p$ be a point in $A$.
Define
$H_p$ = The half-line emanating from the origin that contains $p$.

Definition: Given an Equivalence Relation $R$ on a Set $A$ and given an element $a \in A$, define "The Equivalence Class of $a$" to be the set (denoted by "$[a]$" = "The Class of $a$")

$$[a] = \{\text{all elements } x \in A \mid x R a\};$$

That is, (as a procedural definition)

For all $x \in A$, $x \in [a] \Leftrightarrow x R a$.

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Important Fact:

Given any Equivalence Relation $R$ on set $A$, the collection of all the Equivalence Classes is a Partition of the underlying set $A$.

Problem 1: Let $R$ be the "The Punctured Plane Relation" on the set $A$ = "The Punctured plane"

"Determine the Equivalence Classes of Equivalence Relation $R$".

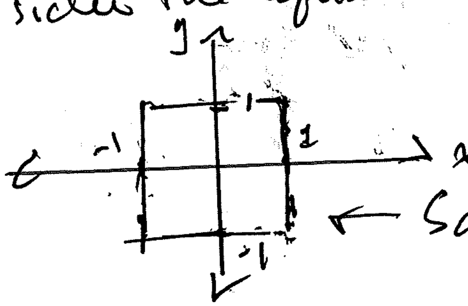
(Translation: Describe the collection of all Equivalence Classes of $R$)

Sol'n 1: Since, for each point $p \in A$,

$$[p] = H_p,$$

The Equivalence Classes of $R$ is $\{$ The Sets $H_p$ for all $p \in A \}$

Sol'n 2: Consider the Square $S$ shown below:




← Square $S$.

The Equivalence Classes of $R$ are $\{$ the sets $H_p$ for all $p \in S \}$

This description describes All the DISTINCT EQUIVALENCE CLASSES of $R$

A description with no redundancies.

The

Ex: "Congruence (mod3)" Relation on $\mathbb{Z}$:

is an equivalence Relation.

For $a, b \in \mathbb{Z}$, $a \equiv b \pmod 3 \iff 3 \mid (a-b)$

$$a \equiv_{(mod 3)} b \iff 3 \mid (a-b).$$

Find $[2]$, a set of integer

$2 \equiv_{(mod 3)} 2$, $2 \in [2]$

$5 \equiv_{(mod 3)} 2$, $5 \in [2]$.

$8 \equiv_{(mod 3)} 2$, $8 \in [2]$

$-1 \equiv_{(mod 3)} 2$. $-1 \in [2]$

$[2] = \{ \ldots, -7, -4, -1, 2, 5, 8, 11, 14, \ldots \} = [8]$

$[1] = \{ \ldots, -8, -5, -2, 1, 4, 7, 10, 13, \ldots \} = [10]$

$[0] = \{ \ldots, -9, -6, -3, 0, 3, 6, 9, 12, \ldots \} = [3] = [9]$

$[0] \cup [1] \cup [2] = \mathbb{Z}$.

The Congruence classes form a Partition of $\mathbb{Z}$.

EXAMPLE PROOF TO HELP WITH PROBLEM #13 of Sec. 8.2

Sec 8.3, #19 (NOT ASSIGNED)

Define the relation $F$ on $\mathbb{Z}$ by requiring that for all $m, n \in \mathbb{Z}$,

$$m F n \iff 4 \mid (m-n).$$

To Prove: $F$ is an equivalence relation

Proof: [F is reflexive]

Let $x$ be any integer. Then, $x - x = 0 = 4 \times 0$.
$\therefore 4 \mid (x-x)$. $\therefore xFx$, by def'n of relation $F$.
$\therefore F$ is reflexive, by direct proof.

[F is symmetric]

Let $x$ and $y$ be any integers. Suppose $xFy$. [NTS: $yFx$].
Then, $4 \mid (x-y)$, by def'n of relation $F$. $\therefore x - y = 4k$ for some integer $k$.
$\therefore (y-x) = (-1)(x-y) = (-1)(4k) = 4(-k)$.
$\therefore 4 \mid (y-x)$. $\therefore yFx$, by def'n of relation $F$.
$\therefore F$ is symmetric, by direct proof.

[F is transitive] Let $x$, $y$ and $z$ be integers such that $xFy$ and $yFz$.
[NTS: $xFz$]. $\therefore$ By def'n of $F$, $4 \mid (x-y)$ and $4 \mid (y-z)$.
$\therefore$ There exist integers $k$ and $l$ such that $(x-y) = 4k$ and $(y-z) = 4l$. Now, $(x-z) = (x-y) + (y-z)$.
$\therefore x - z = 4k + 4l$, by substitution, $\therefore x - z = 4(k+l)$.
$\therefore 4 \mid (x-z)$. $\therefore xFz$, by def'n of relation $F$.
$\therefore F$ is transitive by direct proof.

$\therefore F$ is reflexive, symmetric and transitive. $\therefore F$ is an equivalence relation. QED

# THE PROOF OF LEMMA 3.8.2

Lemma 8.3.2. Let $A$ be a set, and let $R$ be an EQUIVALENCE relation on $A$. [Comm! does not need defn in the Proof]

For all $a, b \in A$, if $aRb$, then $[a] = [b]$.

Proof: Let $a, b \in A$ be given.

Suppose $aRb$. [NTS: $[a] \subseteq [b]$ AND $[b] \subseteq [a]$]

[Showing $[a] \subseteq [b]$]

Let $x \in [a]$ be given. [NTS: $x \in [b]$]

$\therefore$ $xRa$ by def'n of $[a]$.

Since $aRb$ and $R$ is transitive, $xRb$.

$\therefore$ $x \in [b]$ by defn of $[b]$.

$\therefore$ $[a] \subseteq [b]$, by Direct Proof.

[Showing $[b] \subseteq [a]$]

Let $y \in [b]$ be given. [NTS: $y \in [a]$]

$\therefore$ $yRb$ by defn of $[b]$.

$\therefore$ Since $aRb$ and $R$ is symmetric, $bRa$.

$\therefore$ Since $R$ is transitive and $yRb$ and $bRa$, $yRa$.

$\therefore$ $y \in [a]$, by defn of $[a]$.

$\therefore$ $[b] \subseteq [a]$ by Direct proof.

$\therefore$ $[a] = [b]$ by definition of set equality.

$\therefore$ For all $a, b \in A$, if $aRb$, then $[a] = [b]$, by Direct proof. QED